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SUBSPACES OF l_p^N OF SMALL CODIMENSION

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ABSTRACT

In this paper the structure of subspaces and quotients of l_p^N of dimension very close to N is studied, for $1 \leq p \leq \infty$. In particular, the maximal dimension $k = k(p, m, N)$ so that an arbitrary m-dimensional subspace X of l_p^N contains a good copy of l_p^k, is investigated for $m = N - o(N)$. In several cases the obtained results are sharp.

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1. Introduction

One of the most interesting problems in the local theory of Banach spaces is to estimate the maximal dimension $k = k(p, m, N)$ so that an arbitrary m-dimensional subspace X of l_p^N, with $1 \le p \le \infty$, contains a good copy of l_p^k.

This problem has of course an obvious geometric content. For instance, in the case $p = \infty$ and $m < N$, this is the same as estimating the maximal $k = k(\infty, m, N)$ such that a k-dimensional cube can be "embedded" in any m-dimensional section of an N-dimensional cube; while, for $p = 1$, the word "cube" should be replaced by "octahedron".

The cases when m is proportional to N or m is $o(N)$ were studied quite intensively and many results of importance were proved, though the picture is still far from being completely clear. For $1 \le p < 2$ and $m \le cN$ for some fixed $0 < c < 1$, and for $p > 2$ and $m \le N^{2/p}$, and for $p = \infty$ and $m \le \log N$, the function $k(p, m, N)$ remains bounded, since in all these cases, l_p^N contains Euclidean subspaces of dimension m. For $1 \le p < 2$, this fact was proved independently in [F-L-M] and [K.3]. For $p > 2$ this is an immediate consequence of a result from [M] (14); by another approach this was also proved in [B-D-G-J-N] (cf. also [M-S]).

For $p = \infty$ and $N^\delta \le m \le cN$, for some $0 < \delta$ and $c < 1$, it is known that $k(\infty, m, N)$ is of order of magnitude $m^{1/2}$ ([F-J], for m proportional to N and [B1], for m of power type). Recently, $k = k(p, m, N)$ was also calculated for $p > 2$ ([B-T.2]), yielding the estimate $k \ge c \min(m^{p'/2}, (m/N^{2/p})^{p/p-2})$, for some $c > 0$, where $p' = p/(p-1)$. This estimate is sharp.

The aim of the present paper is to complement this line of research by studying the case when m is "very" large, meaning that $m = N - n$ with $n = o(N)$. This is of course the most natural case appearing in analysis and one expects that subspaces of l_p^N of a relatively small codimension inherit more of the structure of the underlying space l_p^N. It turns out that this is indeed the case. For instance, already in the case $p = 1$, a dramatic change of behaviour occurs. While $k(1, m, N)$ remains bounded for m proportional to N, in the present context, $k(1, N-n, N)$ is behaving asymptotically as $\min((N/n)\log(1 + N/n), N)$, which is best possible. In particular, for $n = \log N$, we get the remarkable fact that a subspace of l_1^N of codimension smaller than n contains l_1^k, for k proportional to N.

We are not able to estimate $k(p, N - n, N)$, for $1 < p < 2$ and $n = o(N)$.

This question is related to the recent example of Bourgain [B.2] which shows that $k(p, N-n, N)$ is never proportional to N, for $p \neq 2$, as long as n exceeds a power of N. However, we provide an asymptotically best lower estimate for the type 2 constant and the Euclidean distance of a subspace X of l_p^N, $1 < p < 2$, with $\dim X = N - n$ where $n < N/4$. For example, if $n < N^{2/p'}$, then any such subspace X has already the maximal Euclidean distance, up to a multiplicative constant.

We pass now to the case $p > 2$. In this case, we prove that any $(N-n)$- dimensional subspace X of l_p^N contains a good copy of l_p^k, for k about $(N/n)n^{p'/2}$ (provided $n \leq N/16$). This result, which is not the best possible, is proved by using a random selection method developed in [B-T.2] together with a suitable change of density. Finally, in the case $p = \infty$, we estimate the function $k(\infty, N-n, N)$ in terms of some Gelfand numbers. While this estimate is sharp, its use depends on the possibility of calculating the Gelfand numbers appearing there. For $n \leq \log N$ or n larger than a power of N, precise order of these numbers follows from the results in [K1], [K2], [H] and [G3]. It turns out that, for $n = \log N$, k becomes proportional to N, as in the case of subspaces of l_1^N.

In addition to the above results on "large" subspaces of l_p^N, we have also some results on "large" quotients of l_p^N. In the case, $1 \leq p \leq 2$, any quotient X of l_p^N of dimension $m \geq cN$, for some $c > 0$, contains a good copy of l_p^k, for k already proportional to N ([B-K-T]). We complement these results for the case $p > 2$ and prove that a quotient Y of l_p^N with $\dim Y = N - n$ and $n \leq C(N/\log N)^{2/p}$, for some $C < \infty$, must contain a $(1+\varepsilon)$-isomorphic copy of l_p^k, with k proportional to N. Most likely, the factor $\log N$ can be removed. A corresponding result is proved also for quotients of l_∞^N.

Before we pass to the results described above, some comments on notations are in order. We follow the standard notation in the theory of Banach spaces, *cf.* *e.g.*, [L-T] and [TJ]. In particular, for $1 \leq p \leq \infty$, we consider the real sequence spaces l_p^N, with the norm $\|\cdot\|_p$. For a subset $\sigma \subset \{1, \ldots, N\}$, we set $l_p^\sigma = \{x = (x(i)) \mid x(i) = 0 \text{ for } i \notin \sigma\}$. Sometimes we denote l_p^σ by $l_p^{|\sigma|}$, if the support σ is clear from the context.

For finite-dimensional Banach spaces X and Y of the same dimension, denote by $\mathrm{d}(X, Y)$ the Banach-Mazur distance, i.e., $\inf \|T\| \cdot \|T^{-1}\|$, where the infimum runs over all isomorphisms T from X onto Y. For a finite-dimensional space E, denote by d_E its Euclidean distance $\mathrm{d}(E, l_2^{\dim E})$. Furthermore, for $1 \leq p, q \leq \infty$, denote by $I_{p,q}^N : l_p^N \to l_q^N$ the formal identity operator.

Let us recall the definitions of Gelfand and Kolmogorov numbers. If X and Y are Banach spaces and T is a linear operator from X into Y then, for any n, the n-th *Gelfand number* is defined by

$$c_n(T) = \inf_{\operatorname{codim} E < n} \sup \{ \|Tx\|_Y / \|x\|_X : x \in E \subset X, \}.$$

The n-th *Kolmogorov number* of T is defined by

$$d_n(T) = \inf_{\dim F < n} \sup \{ \inf_{f \in F} \|Tx - f\|_Y : x \in X, \|x\|_X \leq 1 \}.$$

It is easily checked that for any operator T from X into Y we have

$$c_n(T) = d_n(T^*), \tag{1.1}$$

If $X = (\mathbb{R}^N, \|\cdot\|_X)$ and $Y = (\mathbb{R}^N, \|\cdot\|_Y)$, and $I : X \to Y$ is the formal identity operator, then $c_n(I)$ and $d_n(I)$ will be denoted by $c_n(X, \|\cdot\|_Y)$ and $d_n(X, \|\cdot\|_Y)$, respectively.

2. Subspaces of l_p^N, for $1 \leq p < 2$

The following theorem proves a conjecture of V. Milman concerning the dimension $k = k(N, n)$ of a copy of l_1^k that can be embedded in any subspace E of l_1^N of codimension n. It is interesting to point out that k becomes proportional to N when the codimension n is of order of magnitude $\log N$.

THEOREM 2.1: *Let $E \subset l_1^N$ be a subspace with $\dim E = N - n$ for some $n \leq N/2$. There exists an integer k with*

$$k \geq (1/24)^2 \min\left((N/4n)\log(N/2n), N\right)$$

such that E contains a k-dimensional subspace F satisfying $\mathrm{d}(F, l_1^k) \leq 6$.

The proof is based on estimates of Gelfand numbers of certain norms. Fix an integer $1 \leq s \leq N$ and let $|||\cdot|||_s$ be the norm on $\mathbb{R}^N$ defined by

$$|||x|||_s = \max_{|\sigma| = s} \sum_{i \in \sigma} |x(i)| \quad \text{for} \quad x = (x(i)) \in \mathbb{R}^N. \tag{2.1}$$

It was essentially proved in [G-G] that for every n and N and for $s \geq 4n/\log N/n$, one has

$$c_n(l_1^N, |||\cdot|||_s) \geq 1/3. \tag{2.2}$$

The proof of (2.2), or of the equivalent estimate for Kolmogorov numbers, is based on a discussion of the set of extreme points in the dual space $(\mathbb{R}^N, |||\cdot|||_s)^*$ and on a volume comparison argument.

We now pass to the proof of Theorem 2.1.

Proof of Theorem 2.1: Set $s = \max([4n/\log(N/2n)], 1)$. By (2.2), there exist $x_1 \in E$ with $|||x_1|||_s = 1$ and $\|x_1\|_1 \leq 3$. Pick $\sigma_1 \subset \{1, \dots, N\}$ such that $|\sigma_1| = s$ and $\sum_{i\in\sigma_1} |x_1(i)| = 1$. Let σ_1^c be the complement of σ_1 and set $E_2 = E \cap l_1^{\sigma_1^c}$ considered as a subspace of $l_1^{\sigma_1^c}$.

Note that $|\sigma_1^c| = N - s > N/2$, and so, using (2.2) again,

$$c_{n+1}(l_1^{\sigma_1^c}, |||\cdot|||_s) \geq c_{n+1}(l_1^{N/2}, |||\cdot|||_s) > 1/3.$$

Since $\operatorname{codim} E_2 < n$, there exist $x_2 \in E_2$ with $|||x_2|||_s = 1$ and $\|x_2\|_1 \leq 3$.

Continuing this way, we construct by induction a sequence of vectors $x_1, \dots, x_{k'}$ in E, with $k' = [N/2s]$, and $\sigma_1, \dots, \sigma_{k'}$, mutually disjoint subsets of $\{1, \dots, N\}$, such that $|\sigma_j| = s$ and $x_{j|\bigcup_{i=1}^{j-1}\sigma_i} = 0$, for $j = 1, \dots, k'$. Moreover, for $j = 1, \dots, k'$ one has

$$\|x_{j|\sigma_j}\|_1 = \sum_{i\in\sigma_j} |x_j(i)| = 1 \quad \text{and} \quad \|x_j\|_1 \leq 3.$$

By Schechtman's argument (cf. [J-S]), there is a subset $\eta \subset \{1, \dots, k'\}$ with $|\eta| \geq k'/(3 \cdot 2^5)$ such that

$$\sum_{\substack{i\in\eta \\ i\neq j}} \|x_{j|\sigma_i}\|_1 \leq 1/2 \quad \text{for} \quad j \in \eta.$$

Set $F = \operatorname{span}[x_j]_{j\in\eta}$. A well-known perturbation argument shows that for any sequence of scalars $(c_j)_{j\in\eta}$ we have

$$3\sum_{j\in\eta} |c_j| \geq \Big\|\sum_{j\in\eta} c_j x_j\Big\|_1 \geq (1/2)\sum_{j\in\eta} |c_j|.$$

Finally, $\dim F \geq a(N/n)\log(1 + N/n)$, for some $a \geq 2^{-6}3^{-2}$. ∎

Remark: It would be interesting to know the following nearly isometric version of Theorem 2.1. Given $\epsilon > 0$, what is the largest dimension $k = k(N, n, \epsilon)$ such that every $(N-n)$-dimensional subspace of l_1^N contains a k-dimensional subspace $(1+\epsilon)$-isomorphic to l_1^k? In particular, is k proportional to N if n is of order $\log N$? A proof of this fact would obviously follow from a refined version of (2.2).

COROLLARY 2.2: *Under the assumptions of Theorem 2.1, we have*

$$\mathrm{d}_E \geq (1/24)\min\left((N/4n)\log(N/2n), N\right)^{1/2}.$$

On the other hand, a random $(N-n)$-dimensional subspace $E \subset l_1^N$ satisfies

$$\mathrm{d}_E \leq C(N/n)^{1/2}(\log(1+N/n))^{1/2},$$

for some universal constant $C \geq 1$. In particular, whenever $F \subset E$ is a k-dimensional subspace satisfying $\mathrm{d}_F \leq 2$, then $k \leq 4C(N/n)\log(1+N/n)$.

Proof: The first part follows from Theorem 2.1. The second is a consequence of the result in [G-G] where it was proved that for a random $(N-n)$-dimensional subspace $E \subset l_1^N$, the restriction of the formal identity operator $I_{1,2}$ satisfies

$$\|I_{1,2|E}\| \leq C\min\left((1/n)^{1/2}(\log(1+N/n))^{1/2}, 1\right).$$

for some universal constant $C \geq 1$. ∎

The case $1 < p < 2$ is considerably more difficult than that of $p = 1$. The result below provides a lower estimate for the type 2 constant and hence for the Euclidean distance of a subspace E of l_p^N, in terms of the codimension of E. Since by [L] $T_2(E) \leq \mathrm{d}_E \leq (\dim E)^{1/p-1/2}$, it follows in particular that the type 2 constant is of maximal order whenever $\operatorname{codim} E \leq N^{2/p'}$, with $p' = p/(p-1)$.

THEOREM 2.3: *Let $1 < p < 2$. Any subspace E of l_p^N with $\dim E = N - n$ for some $n \leq N/4$ has the type 2 constant satisfying*

$$T_2(E) \geq (1/2)\min((N/n)^{1/2}, N^{1/p-1/2}). \tag{2.3}$$

In particular, $\mathrm{d}_E \geq (1/2)\min((N/n)^{1/2}, N^{1/p-1/2})$. On the other hand, for a random $(N-n)$-dimensional subspace E, one has

$$\mathrm{d}_E \leq C\min((N/n)^{1/2}, N^{1/p-1/2}),$$

where C is a universal constant.

Proof: Let Q be orthogonal projection onto E and $P = I - Q$. Then

$$\Big(\sum_{j=1}^{N} \|Pe_j\|_p^2\Big)^{1/2} \le N^{1/p-1/2}\Big(\sum_{j=1}^{N} \|Pe_j\|_2^2\Big)^{1/2}$$
$$= N^{1/p-1/2}\mathrm{hs}(P) = N^{1/p-1/2}n^{1/2},$$

where hs(P) denotes the Hilbert-Schmidt norm of the operator P. It follows that

$$\Big(\sum_{j=1}^{N} \|Qe_j\|_p^2\Big)^{1/2} \le N^{1/2} + N^{1/p-1/2}n^{1/2}.$$

On the other hand, since

$$\Big(\int \|\sum_{j=1}^{N} \varepsilon_j Pe_j\|_2^2 \, d\varepsilon\Big)^{1/2} = \Big(\sum_{j=1}^{N} \|Pe_j\|_2^2\Big)^{1/2} = \mathrm{hs}(P) = n^{1/2},$$

then

$$\Big(\int \|\sum_{j=1}^{N} \varepsilon_j Qe_j\|_p^2 \, d\varepsilon\Big)^{1/2} \ge N^{1/p} - N^{1/p-1/2}n^{1/2} \ge N^{1/p}/2,$$

whenever $n \le N/4$. Thus,

$$T_2(E) \ge (1/2)\min\big((N/n)^{1/2}, N^{1/p-1/2}\big),$$

as required.

The statement about random subspaces is a consequence of a result from [G2]. It is proved there that for a random $(N-n)$-dimensional subspace E, the restriction to E of the formal identity operator $I_{p,2}$ from l_p^N to l_2^N satisfies

$$\|I_{p,2|E} : E_p \to l_2^N\| \le C \min\big(N^{1/p'}n^{-1/2}, 1\big),$$

for some universal constant C. Since $\|I_{p,2}^{-1}\| \le N^{1/p-1/2}$, this shows the upper estimate for the distance $d(E, l_2^{N-n})$ in this case. ∎

Remark 1: Theorem 2.3 implies the lower estimate for Gelfand numbers $c_n(l_p^N, l_2^N)$ $(1 < p < 2)$ obtained in [G1]. In particular, our argument provides an essentially simpler alternative proof of this estimate.

Remark 2: Combining an argument similar to the one above with the invertibility result from [B-T.1] we can show that under the assumptions of Theorem 2.3, there is $k \geq a \min((N/n)^{p/(2-p)}, N)$ (where $a > 0$ is a universal constant) such that the subspace E contains vectors $x_1 \ldots, x_k$ satisfying

$$a\Big(\sum_{i=1}^{k} |t_i|^p\Big)^{1/p} \leq \Big\|\sum_{i=1}^{k} t_i x_i\Big\|_p \leq \sum_{i=1}^{k} |t_i|, \tag{2.4}$$

for every $(t_i) \in \mathbb{R}^k$.

Remark 3: Let us recall the long standing conjectures that if E is an N-dimensional subspace of L_p whose Euclidean distance is of maximal order, or E is a subspace of l_p^N with codimension $N^{2/p'}$, then E must contain a copy of l_p^k with k proportional to N. It has been recently shown by Bourgain in [B2] that these conjectures are false. Bourgain constructed, for each value of $0 < \varepsilon < 1$ and N, a subspace E of l_p^N of codimension $n \leq N^\varepsilon$ which does not contain good copies of l_p^k, for k proportional to N. By Theorem 2.3, this subspace also has the Euclidean distance of maximal order.

3. Subspaces and quotients of l_p^N, for $2 < p < \infty$

Our first result is concerned with the dimension k of good copies of l_p^k that are contained in subspaces of l_p^N of small codimension, for $p > 2$. For technical reasons we require the codimension n not to exceed $N/16$. This complements the result from [B-T.2] concerning subspaces of l_p^N of (small) dimension $m = N - n$, where estimates of the right order of magnitude were obtained for those values of m which are not considered here.

THEOREM 3.1: *Let $p > 2$ and let*

$$k = [(N/n)n^{p'/2}]/8.$$

Then every subspace E of l_p^N with $\dim E = N - n$ and $n < N/16$ contains a k-dimensional subspace $F \subset E$ such that $\mathrm{d}(F, l_p^k) \leq C$, where $C = C(p) < \infty$ depends on p only.

The proof of the theorem is based on a modification of the supression theorem from [B-T.2], Corollary 2.4. In order to describe this modification, fix numbers n, N and K, with $n \leq \min(K, N)$, and consider systems of vectors $\{\varphi_i\}_{i=1}^K$ in an $L_p(\mu)$-space, for $p > 2$ and some probability measure μ, which satisfy the following conditions:

$$\left\| \sum_{i=1}^{K} a_i \varphi_i \right\|_{L_p} \leq n^{1/2-1/p} \|a\|_p, \tag{3.1}$$

for any sequence $a = (a_1, a_2, \dots, a_K) \in l_p^K$,

$$\left\| \left(\sum_{i=1}^{K} |\varphi_i|^2 \right)^{1/2} \right\|_\infty \leq N^{1/p}, \tag{3.2}$$

$$\|\varphi_i\|_{L_p} \leq 1, \quad \text{for all} \quad 1 \leq i \leq K. \tag{3.3}$$

Set

$$\delta = \min \left(n^{-p'(1/2-1/p)}, \left(\frac{K}{N}\right)^{2/(p-2)}\right). \tag{3.4}$$

Then the same proof as in [B-T.2], 2.1, 2.2 and 2.3 shows the following:

THEOREM 3.2: *For every $p > 2$ there exists a constant $D = D(p) < \infty$ such that, whenever $\{\varphi_i\}_{i=1}^K$ is a sequence of elements in an $L_p(\mu)$-space, for some probability measure μ, which satisfies the conditions (3.1), (3.2) and (3.3) above, then a random subset $\sigma \subset \{1, 2, \dots, K\}$ of cardinality*

$$k = [\delta K],$$

where δ is defined by (3.4), contains in turn a subset σ_0 of cardinality $|\sigma_0| > k/2$ such that

$$\left\| \sum_{i \in \sigma_0} a_i \varphi_i \right\|_{L_p} \leq D \|a\|_p,$$

for any sequence $a = (a_i)_{i \in \sigma_o} \in l_p^{|\sigma_o|}$.

We also require a change of density procedure well-known to specialists (see e.g. [B-K-T], [B-L-M], [B-T.2]), which goes back to Lewis [L]. It is stated for subspaces of the space $L_q(\nu) = L_q^N(\nu)$, where ν is the uniform probability measure on $\{1, \dots, N\}$. Let μ be another probability measure on $\{1, \dots, N\}$ such that $\mu(k) > 0$, for every $1 \leq k \leq N$. The operator I_μ from $L_q(\nu)$ to $L_q(\mu)$, defined by

$$(I_\mu h)(j) = h(j)[\nu(j)/\mu(j)]^{1/q},$$

for $j = 1, \dots, N$ and $h \in L_q(\nu)$, is an isometry from $L_q(\nu)$ onto $L_q(\mu)$.

LEMMA 3.3: *Let $1 \le q < \infty$ and $0 < \beta < 1$. Let $X \subset L_q^N(\nu)$ be an n-dimensional subspace. There exists a probability measure μ on $\{1,\dots,N\}$ such that $\mu(j) \ge \beta/N$, for every $j = 1,\dots,N$, which has the property*

$$\|x\|_{L_\infty(\mu)} \le A\sqrt{n}\ \|x\|_{L_2(\mu)}, \quad \text{for} \quad x \in \tilde{X} = I_\mu X, \tag{3.5}$$

where $A \le \max\left((1-\beta)^{-1}, 2(1-\beta)^{-1/2}\right)$. Moreover, for any $p \ge 2$ and $x \in I_\mu$,

$$\|x\|L_p(\mu) \le \left(A\sqrt{n}\right)^{1-2/q} \|x\|L_2(\mu). \tag{3.6}$$

Proof of Theorem 3.1: Suppose that E is an $(N-n)$-dimensional subspace of $L_p^N(\nu)$, for some $n < N/16$. Recall that ν is the uniform measure on $\{1,\dots,N\}$ and $p > 2$. Let $X = E^\perp \subset L_{p'}^N(\nu)$ be the annihilator of E. Apply Lemma 3.3 to X with $q = p' < 2$ and $\beta = 1/2$, so that $A(\beta) \le 2^{3/2}$. Let μ be the corresponding measure on $\{1,\dots,N\}$ and $\tilde{X} = I_\mu X \subset L_{p'}^N(\mu)$ the isometric image of X, determined by Lemma 3.3. Clearly, E^* is isometric to the quotient space $L_{p'}^N(\mu)/\tilde{X}$, hence E is isometric to some subspace E_1 of $L_p^N(\mu)$. In fact, $E_1 = \tilde{X}^\perp$, with the annihilator understood in the duality of $L_{p'}^N(\mu)$ and $L_p^N(\mu)$; hence, also $\tilde{X} = (E_1)^\perp$. Thus, without loss of generality, we may assume that the given space E is already E_1.

We pass now to the construction. Let $e_1,\dots,e_N$ denote the standard unit vector basis in $\mathbb{R}^N$. Let P be the orthogonal projection from $L_2(\mu)$ onto X and set

$$\eta = \{1 \le i \le N \mid \mu(\{i\}) \le 2/N\}.$$

Since μ is a probability measure it follows that $|\eta| \ge N/2$.

In the main step of the proof we show that there exists $\sigma \subset \eta$ with $|\sigma| \ge N/4$ such that the vectors

$$x_i = N^{1/p} P e_i,$$

with $i \in \sigma$, satisfy the hypotheses of Theorem 3.2.

Indeed, by (3.6) we have, for every $x \in L_p^N(\mu)$,

$$\begin{aligned}
\|P_x\|L_p(\mu) &\le (2^3 n)^{1/2-1/p} \|Px\|_{L_2(\mu)} \\
&\le (2^3 n)^{1/2-1/p} \|x\|_{L_2(\mu)} \\
&\le (2^3 n)^{1/2-1/p} \|x\|_{L_p(\mu)}.
\end{aligned}$$

Hence, for any sequence $a = (a_i)_{i\in\eta} \in l_p^{|\eta|}$, we have

$$(3.7)\quad \Big\|\sum_{i\in\eta} a_i x_i\Big\|_{L_p(\mu)} \le N^{1/p}(2^3 n)^{1/2-1/p}\Big\|\sum_{i\in\eta} a_i e_i\Big\|_{L_p(\mu)} \le 2^{3/2} n^{1/2-1/p}\,\|a\|_p.$$

Next, by (3.5), we get that

$$\begin{aligned}\Big\|(\sum_{i\in\eta}|x_i|^2)^{1/2}\Big\|_{L_\infty(\mu)} &= \sup\{\Big\|\sum_{i\in\eta} s_i x_i\Big\|_{L_\infty(\mu)} : \sum s_i^2 \le 1\}\\ &\le 2^{3/2} n^{1/2}\sup\{\Big\|\sum_{i\in\eta} s_i x_i\Big\|_{L_2(\mu)} \mid \sum s_i^2 \le 1\}\\ &\le 4n^{1/2} N^{1/p-1/2},\end{aligned}$$

which, since $n \le N$, implies that

$$(3.8)\qquad \Big\|(\sum_{i\in\eta}|x_i|^2)^{1/2}\Big\|_{L_\infty(\mu)} \le 4N^{1/p}.$$

Finally,

$$\begin{aligned}(\sum_{i\in\eta}\|x_i\|_{L_2(\mu)}^2)^{1/2} &\le N^{1/p}(\sum_{i\in\eta}\|Pe_i/\mu(i)^{1/2}\|_{L_2(\mu)}^2)^{1/2}(2/N)^{1/2}\\ &\le 2^{1/2} N^{1/p-1/2}\,\mathrm{hs}\,(P) \le (2n)^{1/2} N^{1/p-1/2}.\end{aligned}$$

Therefore, for some subset $\sigma' \subset \eta$ with $|\sigma'| \ge |\eta|/2 \ge N/4$, for all $i \in \sigma'$ we get $\|x_i\|_{L_2(\mu)} \le (2^3 n)^{1/2} N^{-1/p'}$. Since $n \le N/16$, this yields by (3.6) that

$$(3.9)\qquad \|x_i\|_{L_p(\mu)} \le (2^3 n)^{1/2-1/p}\,\|x_i\|_{L_2(\mu)} \le (2^3 n/N)^{1/p'} \le 2^{-1/p'} < 1,$$

for all $i \in \sigma'$.

By (3.7), (3.8), (3.9) and by Theorem 3.2 for $K = |\sigma'|$, we get that for $\delta = n^{-p'(1/2-1/p)}$, there is $\sigma \subset \sigma'$ with

$$(3.10)\qquad |\sigma| = \delta|\sigma'|/2 \ge (N/n) n^{p'/2}/8$$

such that

$$(3.11)\qquad \Big\|\sum_{i\in\sigma} a_i x_i\Big\|_{L_p(\mu)} \le D\,\|a\|_p,$$

for any sequence $a = (a_i)_{i\in\sigma} \in l_p^{|\sigma|}$.

In order to conclude the proof observe that, for $i \in \eta$,

$$\|e_i\|_{L_p(\mu)} = \mu(\{i\})^{1/p} \ge (\beta/N)^{1/p} \ge (2N)^{-1/p}.$$

Therefore, the vectors $z_i = N^{1/p}e_i - x_i$, with $i \in \sigma$, satisfy

$$\|z_i\|_{L_p(\mu)} \geq (2^{1/p'} - 2^{1/p})/2, \quad \text{for} \quad i \in \sigma. \tag{3.12}$$

Also, by (3.11), we get that

$$\Big\|\sum_{i\in\sigma} a_i z_i\Big\|_{L_p(\mu)} \leq N^{1/p}\Big\|\sum_{i\in\sigma} a_i e_i\Big\|_{L_p(\mu)} + \Big\|\sum_{i\in\sigma} a_i x_i\Big\|_{L_p(\mu)} \leq (D + 2^{1/p})\|a\|_p,$$

for any sequence $a = (a_i)_{i\in\sigma} \in l_p^{|\sigma|}$, where D is the constant from the statement of Theorem 3.2.

Combining the latter upper p-estimate with (3.12) we get, by Corollary 3.10 from [B-T.1], that there exists $\sigma'' \subset \sigma'$ such that $|\sigma''| > c|\sigma|$, for some $c > 0$, and $\text{span}\,(z_i)_{i\in\sigma''}$ is C-isomorphic to $l_p^{|\sigma''|}$, for some constant $C = C(p)$. Since $z_i = N^{1/p}(I - P)e_i \in E$, for all i, this completes the proof in view of (3.10). ∎

Contrary to the case of subspaces of dimension n, with $n \leq N/2$, considered in [B-T.2], the formula for k obtained in Theorem 3.1 is not the best possible. The recent example of Bourgain [B2], already mentioned in the previous section, shows that the maximal k for which Theorem 3.1 holds cannot be proportional to N, whenever n is a power of N.

We pass now to the investigation of quotients of l_p^N, for $p > 2$. It already follows from Theorem 2.3 that if X is a quotient of l_p^N with $\dim X = N - n$, where $n \leq CN^{2/p}$, then the Euclidean distance of X is of maximal order, i.e., $\mathrm{d}_X \geq aN^{1/2-1/p}$, where $C > 0$ is a universal constant and $a = a(C) > 0$. However, contrary to the dual situation of subspaces of l_p^N discussed in the earlier section, such a quotient space X most likely contains a subspace of dimension k proportional to N which is isomorphic to l_p^k. We prove this fact under a slightly stronger assumption that $n \leq C(N/\log N)^{2/p}$.

THEOREM 3.4: *For every $2 < p < \infty$ and $\varepsilon > 0$, there exists a constant $C = C(p,\varepsilon) > 1$ such that, whenever $n \leq C(N/\log N)^{2/p}$ and Y is a quotient of l_p^N of dimension $N-n$, then Y contains a subspace $F \subset Y$ with $k = \dim F \geq N/C$ such that $\mathrm{d}(F, l_p^k) \leq 1 + \varepsilon$.*

The proof of the theorem requires the following analytic fact:

PROPOSITION 3.5: *For every $2 < p < \infty$ and $0 < a < 1$, there exists a constant $c(p,a) > 0$ such that, whenever $X \subset l_p^N$ is an n-dimensional subspace, with n satisfying $N/\log N \geq c(p,a)n^{p/2}$, then one can find a subset $\sigma \subset \{1,\dots,N\}$ with $|\sigma| \geq [(a/6)N]$ such that*

$$\|R_\sigma x\|_p \leq a\|x\|_p \qquad \text{for all } x \in X, \tag{3.13}$$

where R_σ denotes the restriction operator to σ.

The proof of the proposition uses techniques developed in [B-M-L]; it is based on a random approach combined with entropy estimates.

Let us recall first that, for two convex subsets B and $\tilde{B}$ of a linear space and for $t > 0$, the covering number $E(B,\tilde{B},t)$ is defined by

$$E(B,\tilde{B},t) = \min\{M : \exists \{x_i\}_{i=1}^M \quad \text{so that} \quad \tilde{B} \subset \bigcup_{1 \leq M} (x_i + tB)\}.$$

We will require upper estimates for some covering numbers in the case of unit balls in subspaces of l_p^N. In [B-M-L] it was proved that, for any $0 < \beta < 1$, there exists a constant $A' = A'(\beta)$ so that, whenever $X \subset l_p^N$ is an n-dimensional subspace, μ is the probability measure on $\{1,\dots,N\}$ given by Lemma 3.3 which determines the isometry I_μ and satisfies (3.5) above, and B_p and B_∞ denote the unit balls in $I_\mu(X) \subset L_p^N(\mu)$ and in $I_\mu(X) \subset L_\infty^N(\mu)$ respectively, then, for all $t > 0$, we have that

$$\log E(B_p, B_\infty, t) \leq A'pnt^{-2}\log N. \tag{3.14}$$

Clearly, by increasing A in (3.5), if necessary, we may assume that both estimates (3.5) and (3.14) are valid with the same constant $A(\beta)$.

The following lemma is proved in detail in [B-L-M] (see the proof of Theorem 7.3 there).

LEMMA 3.6: *For every $2 \leq p < \infty$, $b > 1$, $0 < \delta < 1$ and $A \geq 1$, there exists a constant $c'(p,b,\delta,A) > 0$ such that if $X \subset L_p^N(\mu)$ is an n-dimensional subspace which satisfies (3.5) and (3.14), with the constant A, and $N/\log N \geq c'n^{p/2}$, then there exists a subset $\sigma \subset \{1,\dots,N\}$ with $M = |\sigma| = [N/b]$ such that*

$$\left|M^{-1}\sum_{j\in\sigma}|x(j)|^p - 1\right| < \delta, \tag{3.15}$$

for every $x = (x(j)) \in X$ with $\|x\|_p = 1$.

Now we are ready for the

Proof of Proposition 3.5: Let $\delta = 1/2$, $\beta = 1 - a/(6-a)$ and $b = 3/a$. Let μ and I_μ be given by Lemma 3.3 so that, in particular, $\mu(j) \geq \beta/N$, for all $j = 1, \dots, N$. Since I_μ is an isometry from l_p^N onto $L_p^N(\mu)$ preserving the lattice structure, it is sufficient to prove the proposition for $I_\mu(X) \subset L_p^N(\mu)$.

Set $c(p,a) = c'(p,b,\delta,A(\beta))$, assume that $N/\log N \geq cn^{p/2}$ and apply Lemma 3.6. So there is a subset $\sigma' \subset \{1,\dots,N\}$ with $M = |\sigma'| = [N/b]$ for which (3.15) is satisfied. Let $J = \{1 \leq j \leq N : \mu(j) \leq 2/N\}$ and let J^c be the complement of J. Set

$$\mu_0 = \sum_{j \in J^c} \mu(j)$$

and notice that $|J|(\beta/N) \leq 1 - \mu_0$ and $|J^c|(2/N) \leq \mu_0$. Thus,

$$N = |J| + |J^c| \leq (1-\mu_0)N/\beta + \mu_0 N/2,$$

which further yields that

$$|J^c| \leq \mu_0 N/2 \leq N(1-\beta)/(2-\beta). \tag{3.16}$$

Set $\sigma = \sigma' \cap J$. By the definition of J and (3.15) we have

$$\sum_{j\in\sigma} \mu(j)|x(j)|^p \leq (2/N)\sum_{j\in\sigma'} |x(j)|^p \leq (2/b)(1+\delta) \leq a, \tag{3.17}$$

for all $x = (x(j)) \in X$ with $\|x\|_p = 1$.

Moreover, by (3.16),

$$|\sigma| \geq |\sigma'| - |J^c| \geq N/b - 1 - N(1-\beta)/(2-\beta) \geq Na/6 - 1.$$

This concludes the proof. ∎

We finally pass to the

Proof of Theorem 3.4: Fix $p > 2$ and $\varepsilon > 0$ and pick $0 < a < 1$ such that $(1-a)/(1+a) = (1+\varepsilon)^{-1}$. Let $c = c(p,a)$ be the constant appearing in the statement of Proposition 3.5 and let Y be a quotient of l_p^N of dimension $N - n$ for which $cn^{p/2} \leq N/\log N$. Denote by Q the quotient map from l_p^N onto Y and set $X = \ker Q$. Then, by Proposition 3.5, there exists a set $\sigma \subset \{1,\dots,N\}$ with $|\sigma| \geq aN/6$ which satisfies (3.13).

Set $E = \operatorname{span}[e_j]_{j\in\sigma} \subset l_p^N$. We will show that $F = Q(E)$ is the required subspace of Y.

Clearly, $k = \dim E = |\sigma| \geq (a/6)N$. Also, for any $(t_j)_{j\in\sigma} \in \mathbb{R}^{|\sigma|}$, one has

$$\left(\sum_{j\in\sigma} |t_j|^p\right)^{1/p} = \|\sum_{j\in\sigma} t_j e_j\|_p \geq \|\sum_{j\in\sigma} t_j(Qe_j)\|_Y. \tag{3.18}$$

Now fix $x = \sum_{j\in\sigma} t_j e_j \in E$ with $\|x\|_p = 1$. Set $\gamma = 2/(1+a) > 1$ and observe that, for $z \in X$ with $\|z\|_p \geq \gamma$, we have

$$\|x - z\|_p \geq \gamma - 1.$$

On the other hand, if $z \in X$ and $\|z\|_p < \gamma$ then, by (4.14), $\|R_\sigma z\|_p < a\gamma$. Since $R_\sigma x = x$ we get that

$$\|x - z\|_p \geq \|R_\sigma(x - z)\|_p \geq \|x\|_p - \|R_\sigma z\|_p > 1 - a\gamma.$$

Therefore,

$$\|Qx\|_Y = \inf_{z\in X} \|x - z\|_p \geq \min(\gamma - 1, 1 - a\gamma).$$

By the choice of γ and a, this yields $\|Qx\|_Y \geq (1+\varepsilon)^{-1}$. Together with (3.18) this shows that $\mathrm{d}(F, l_p^k) \leq 1 + \varepsilon$. ∎

4. Subspaces of l_∞^N

In this section we investigate subspaces of l_∞^N of small codimension. The following theorem provides an asymptotically precise formula for the dimension k of an $(1+\varepsilon)$-isomorphic copy of l_∞^k contained in such a subspace, in terms of some Gelfand numbers. The proof is similar to the construction of Figiel and Johnson [F-J], for $n = N/2$.

THEOREM 4.1: *Let $E \subset l_\infty^N$ be a subspace with $\dim E = N - n$ for some $n \leq N/2$. Let $0 < \varepsilon < 2$. There exists an integer k with*

$$k \geq (\varepsilon/8) N c_{n+1}(l_1^{N/2}, \|\cdot\|_\infty)$$

such that E contains a k-dimensional subspace F satisfying $\mathrm{d}(F, l_\infty^k) \leq 1 + \varepsilon$.

On the other hand, for every $n \leq N/2$ there exists a subspace $E \subset l_\infty^N$ with $\dim E = N - n$, such that whenever F is a subspace of E, say of dimension k, then we have $k \leq \mathrm{d}(F, l_\infty^k) N c_{n+1}(l_1^N, \|\cdot\|_\infty)$.

Proof: Denote $c_{n+1}(l_1^{N/2}, \|\cdot\|_\infty)$ by α. It is easy to see that whenever $N' \geq N/2$, then $c_{n+1}(l_1^{N'}, \|\cdot\|_\infty \geq \alpha$. Therefore, for every $N' \geq N/2$ and any subspace $G \subset \mathbb{R}^{N'}$ with $\operatorname{codim} G \leq n$, there exists $y \in G$ such that

$$\|y\|_\infty = 1 \quad \text{and} \quad \|y\|_1 \leq 1/\alpha. \tag{4.1}$$

Fix $0 < \varepsilon < 2$ and take $0 < \delta < 1$ such that $(1+\delta)/(1-\delta) = 1+\varepsilon$. Let $x_1, \dots, x_k$ be a maximal set of vectors in E such that

$$\|x_j\|_\infty = 1, \qquad \|x_j\|_1 \leq 1/\alpha, \quad \text{for } j = 1, \dots, k,$$
$$\text{and} \qquad \Big\|\sum_{j=1}^k |x_j|\Big\|_\infty \leq 1+\delta. \tag{4.2}$$

It is well-known and easily verified that $F = \operatorname{span}[x_j]_{j=1}^k$ satisfies $\mathrm{d}(F, l_\infty^k) \leq (1+\delta)/(1-\delta)$. Indeed, for any scalar sequence $(a_j)_{j=1}^k$ with $\max_{1\leq j\leq k}|a_j| = |a_{j_0}| = 1$, pick i_0 so that $|x_{j_0}(i_0)| = 1$. Then

$$1+\delta \geq \Big\|\sum_{j=1}^k a_j x_j\Big\|_\infty \geq 1 - |\sum_{j\neq j_0} a_j x_j(i_0)| \geq 1-\delta.$$

To estimate the dimension k of F, let

$$\sigma = \{1 \leq i \leq N : \sum_{j=1}^k |x_j(i)| > \delta\}.$$

Then we have,

$$\delta|\sigma| < \Big\|\sum_{j=1}^k |x_j|\Big\|_1 \leq \sum_{j=1}^k \|x_j\|_1 \leq k/\alpha. \tag{4.3}$$

Assume that $|\sigma| < N/2$ and consider the subspace $G = \{x \in E : x(i) = 0 \text{ for } i \in \sigma\} \subset \mathbb{R}^{|\sigma^c|}$. Clearly, $N' = |\sigma^c| \geq N/2$ and $\operatorname{codim} G \leq n$. Hence, there is an $y \in G$ satisfying (4.1). It follows that the set $\{x_1, \dots, x_k, y\}$ satisfies (4.2), thus contradicting the maximality of the x_j's. Therefore $|\sigma| \geq N/2$ which yields, by (4.3), that

$$k \geq (\delta/2)N\alpha.$$

This completes the first part of the proof. For the converse part, pick a subspace $E \subset \mathbb{R}^N$ with $\dim E = N - n$ such that the restriction of the formal identity map $I_{1,\infty}$ to E satisfies

$$\|I_{1,\infty|E}\| = c_{n+1}(I_{1,\infty}) = c_{n+1}(l_1^N, \|\cdot\|_\infty),$$

and let E_1 (resp. E_∞) denote E when considered as a subspace of l_1^N (resp. l_∞^N). Then, for any subspace $F \subset E_\infty$ fix an isomorphism $T : l_\infty^k \to F$ onto F with $\|T\|\,\|T^{-1}\| = C = \mathrm{d}(F, l_\infty^k)$. Let $S : E_\infty \to l_\infty^k$ be an extension of T^{-1} with $\|S\| = \|T^{-1}\|$. Then the identity operator $id_{l_\infty^k}$ on l_∞^k admits the factorization $i_{l_\infty^k} = S\, I_{1,\infty|E}\, I_{\infty,1|E}\, T$, as follows:

$$l_\infty^k \xrightarrow{T} E_\infty \xrightarrow{I_{\infty,1|E}} E_1 \xrightarrow{I_{1,\infty|E}} E_\infty \xrightarrow{S} l_\infty^k.$$

Therefore,

$$\begin{aligned} k &= \operatorname{tr} id_{l_\infty^k} \le \pi_1(id_{l_\infty^k}) \\ &\le \|T\|\,\pi_1(I_{\infty,1|E})\,\|I_{1,\infty|E}\|\,\|S\| \le CNc_{n+1}(l_1^N, \|\cdot\|_\infty), \end{aligned}$$

thus completing the proof. ∎

Let us recall some known estimates for the Gelfand numbers $c_n(l_1^N, l_\infty^N)$. The upper estimate was obtained in [K1], [K2], [H] and it states that

$$(4.4)\quad c_n(l_1^N, \|\cdot\|_\infty) \le C \min\left(1, (\log N/n)^{1/2}, (\log N/\log(1+n))n^{-1/2}\right),$$

where C is a universal constant. The lower estimate, which was proved in [G3], states that, for some constant $a > 0$, one has

$$(4.5)\quad c_n(l_1^N, \|\cdot\|_\infty) \ge \begin{cases} an^{-1/2}\left(\frac{\log N}{\log(1+n/\log N)}\right)^{1/2} & \text{if } n \ge \log N \\ a & \text{if } 1 \le n \le \log N \end{cases}$$

Observe that in the ranges $n \le \log N$ and $N^\delta \le n \le N/2$, for some $\delta > 0$, the upper and the lower estimates (4.4) and (4.5) coincide (up to constants depending on δ), thus giving the right order of growth of $c_n(l_1^N, l_\infty^N)$ in these cases. In fact,

$$\begin{array}{rclcll} a & \le & c_n(l_1^N, \|\cdot\|_\infty) & \le & C & \text{for} \quad n \le \log N \\ a(1/\delta)^{1/2} n^{-1/2} & \le & c_n(l_1^N, \|\cdot\|_\infty) & \le & C(1/\delta)n^{-1/2} & \text{for} \quad n \ge N^\delta. \end{array}$$

It follows that subspaces of l_∞^N of codimension $n \le \log N$ contain $(1+\varepsilon)$-isomorphic copies of l_∞^k for k proportional to N. On the other hand, in the power type range of codimension, $N^\delta \le n \le N/2$, the maximal dimension of an $(1+\varepsilon)$-isomorphic copy of l_∞^k contained in the subspace is of the order $(\varepsilon/\delta)Nn^{-1/2}$ (with $0 < \varepsilon < 2$). This gives a counterpart to the results proved by Figiel and Johnson [F-J] for subspaces of l_∞^N of dimension proportional to N, and by Bourgain [B1] for subspaces of power type dimension.

For quotients of l_∞^N we have the following fact which is an immediate consequence of Theorem 2.1.

COROLLARY 4.2:

Let X be a quotient of l_∞^N with $\dim X = N-n$. There exist universal constants $0 < a < 1$ and $C \geq 1$, and an m-dimensional subspace $Y \subset X$ such that $m \geq a(N/n)\log(N/n)$ and $\mathrm{d}(Y, l_1^m) \leq C$.

Added in proof: Nigel Kalton recently informed us that the following almost isometric version of Theorem 2.1 is true. Given $\epsilon > 0$, there is $k = k(N, n, \epsilon) \geq (1/2)(N - (1-\epsilon^{-1})^n)$ such that every $(N-n)$-dimensional subspace $E \subset l_1^N$ contains a k-dimensional subspace $(1+\epsilon)$-isomorphic to l_1^k. In particular, $n \leq (1/2)(\log(1-\epsilon^{-1}))^{-1}\log N$, then $k(N, n, \epsilon) \geq (1/2)(N - N^{1/2})$, which tends to $N/2$ as $N \to \infty$. On the other hand, $k(N, n, \epsilon) \leq N/2$, for all $\epsilon > 0$ and $1 \leq n \leq N$.

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